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On the Eneström–Kakeya Theorem, II

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1. INTRODUCTION AND STATEMENT OF RESULTS

The following result is well known in the theory of the distribution of zeros of polynomials.

THEOREM A (Eneström–Kakeya). *If*

$$a_n \geqslant a_{n-1} \geqslant a_{n-2} \geqslant \cdots \geqslant a_1 \geqslant a_0 > 0, \tag{1.1}$$

then, for $|z| > 1$, $\sum_{k=0}^{n} a_k z^k \neq 0$.

There already exist in the literature [1; 3, Theorems 1–4; 5, Theorem 3; 6] some extensions of the Eneström–Kakeya theorem. Govil and Rahman [3, Theorems 2, 4] generalized this theorem to polynomials with complex coefficients, first by considering the moduli of the coefficients to be monotonically increasing and then by assuming the real parts of the coefficients to be monotonically increasing.

While refining the results of Govil and Rahman [3, Theorems 2, 4], we [2] proved the following

THEOREM B. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a polynomial with complex coefficients such that*

$$|\arg a_k - \beta| \leqslant \alpha \leqslant \pi/2, \qquad k = 0, 1,..., n,$$

for some real β, *and*

$$|a_n| \geqslant |a_{n-1}| \geqslant \cdots \geqslant |a_1| \geqslant |a_0| > 0; \tag{1.2}$$

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then $p(z)$ has all its zeros in the ring-shaped region given by

$$\frac{1}{R^{n-1}[2R(|a_n|/|a_0|) - (\cos\alpha + \sin\alpha)]} \leqslant |z| \leqslant R,$$

where

$$R = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{k=0}^{n-1}|a_k|.$$

THEOREM C. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$. *If* $\operatorname{Re} a_k = \alpha_k$, $\operatorname{Im} a_k = \beta_k$, *for* $k = 0, 1, 2,..., n$, *and*

$$\alpha_n \geqslant \alpha_{n-1} \geqslant \cdots \geqslant \alpha_1 \geqslant \alpha_0 \geqslant 0, \qquad \alpha_n > 0,$$

then $p(z)$ has all its zeros in the ring-shaped region given by

$$\frac{|a_0|}{R_1^{n-1}[2R_1\alpha_n + R_1|\beta_n| - (\alpha_0 + |\beta_0|)]} \leqslant |z| \leqslant R_1,$$

where

$$R_1 = 1 + \frac{1}{\alpha_n}\left[2\sum_{k=0}^{n-1}|\beta_k| + |\beta_n|\right].$$

In this paper, we have sharpened Theorems B and C. More precisely, we prove

THEOREM 1. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ $(\not\equiv 0)$ *be a polynomial with complex coefficients such that*

$$|\arg a_k - \beta| \leqslant \alpha \leqslant \pi/2, \qquad k = 0, 1,..., n,$$

for some real β, and

$$|a_n| \geqslant |a_{n-1}| \geqslant \cdots \geqslant |a_1| \geqslant |a_0|;$$

then $p(z)$ has all its zeros in the ring-shaped region given by

$$R_3 \leqslant |z| \leqslant R_2.$$

Here

$$R_2 = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{1/2},$$

and

$$R_3 = \frac{1}{2M_2^2}\,[-R_2^2|b|(M_2 - |a_0|) + \{4|a_0|R_2^2M_2^3 + R_2^4|b|^2(M_2 - |a_0|)^2\}^{1/2}],$$

where

$$M_1 = |a_n| R,$$

$$M_2 = |a_n| R_2^n \left[R + R_2 - \frac{|a_0|}{|a_n|}(\cos\alpha + \sin\alpha)\right],$$

$$c = |a_n - a_{n-1}|,$$

$$b = a_1 - a_0, \tag{1.3}$$

and R is as defined in Theorem B.

THEOREM 2. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$. *If* $\operatorname{Re} a_k = \alpha_k$, $\operatorname{Im} a_k = \beta_k$, *for* $k = 0, 1, \ldots, n$, *and*

$$\alpha_n \geqslant \alpha_{n-1} \geqslant \alpha_{n-2} \geqslant \cdots \geqslant \alpha_1 \geqslant \alpha_0 \geqslant 0, \qquad \alpha_n > 0,$$

then p(z) has all its zeros in the ring-shaped region given by

$$R_5 \leqslant |z| \leqslant R_4.$$

Here

$$R_4 = \frac{c}{2}\left(\frac{1}{\alpha_n} - \frac{1}{M_3}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{\alpha_n} - \frac{1}{M_3}\right)^2 + \frac{M_3}{\alpha_n}\right\}^{1/2},$$

$$R_5 = \frac{1}{2M_4^2}\,[-R_4^2\,|b|\,(M_4 - |a_0|)$$
$$+ \{4\,|a_0|\,R_4^2 M_4^3 + R_4^4\,|b|^2 (M_4 - |a_0|)^2\}^{1/2}],$$

where

$$M_3 = \alpha_n R_1,$$

$$M_4 = R_4^n[(\alpha_n + |\beta_n|)\,R_4 + \alpha_n R_1 - (\alpha_0 + |\beta_0|)],$$

$$c = |a_n - a_{n-1}|,$$

$$b = a_1 - a_0,$$

and R_1 *is as in Theorem* C.

As remarked earlier, Theorems 1 and 2 are respectively the refinements of Theorems B and C. For the sake of completeness we shall verify that Theorem 1 sharpens Theorem B and for this, we shall prove that

$$R \geqslant R_2 \tag{1.4}$$

and

$$\frac{1}{R^{n-1}[2R(|a_n|/|a_0|) - (\cos\alpha + \sin\alpha)]} \leqslant R_3. \tag{1.5}$$

For this, note that

$$R = \frac{M_1}{|a_n|} \geqslant \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{1/2},$$

if

$$2M_1^2 \geq c(M_1 - |a_n|) + \{c^2(M_1 - |a_n|)^2 + 4M_1^3 |a_n|\}^{1/2},$$

which is true if

$$(M_1 - c)(M_1 - |a_n|) \geq 0. \tag{1.6}$$

Since (1.6) obviously holds, (1.4) follows. To show (1.5) we have obviously from (1.4)

$$\frac{1}{R^{n-1}[2R(|a_n|/|a_0|) - (\cos\alpha + \sin\alpha)]}$$
$$\leqslant \frac{1}{R_2^{n-1}[(R + R_2)(|a_n|/|a_0|) - (\cos\alpha + \sin\alpha)]}$$
$$= \frac{|a_0| R_2}{M_2}. \tag{1.7}$$

Hence it is sufficient to show that

$$\frac{|a_0| R_2}{M_2} \leqslant R_3. \tag{1.8}$$

Now (1.8) holds if

$$(R_2 |b| - M_2)(M_2 - |a_0|) \leqslant 0. \tag{1.9}$$

As (1.9) is evidently true, (1.8) follows. The fact that Theorem 2 is a refinement of Theorem C can be proved on similar lines and we omit the proof.

In general Theorems 1 and 2 give better results than Theorems B and C, but in some cases the results obtained by Theorems 1 and 2 are significantly better than those obtained respectively from Theorems B and C. To illustrate this, we consider

$$p(z) = 2z^5 + 2^{1/2}(1 + i) z^4 + 3^{1/2}iz^3 + (-1 + i) z^2 + (1 + i)z - 1;$$
$$\alpha = \pi/2; \qquad \beta = \pi/2.$$

By Theorem B, we get that all the zeros of $p(z)$ are contained in the region $5.6017 \times 10^{-6} \leqslant |z| \leqslant 8.5605$, while Theorem 1 gives that all the zeros of $p(z)$ are contained in $33925 \times 10^{-6} \leqslant |z| \leqslant 3.2833$.

2. Lemmas

LEMMA 1. *If* $|\arg a_k - \beta| \leqslant \alpha \leqslant \pi/2$, $|\arg a_{k-1} - \beta| \leqslant \alpha$, *and* $|a_k| \geq |a_{k-1}|$, *then*

$$|a_k - a_{k-1}| \leqslant \{(|a_k| - |a_{k-1}|)\cos\alpha + (|a_k| + |a_{k-1}|)\sin\alpha\}.$$

Lemma 1 is due to Govil and Rahman [3].

LEMMA 2. *If $f(z)$ is analytic inside and on the unit circle, $|f(z)| \leqslant M$ on $|z| = 1$, $f(0) = a$, where $|a| < M$, then*

$$|f(z)| \leqslant M \frac{M|z| + |a|}{|a||z| + M}$$

for $|z| < 1$.

Lemma 2 is a well-known generalization of Schwarz's lemma.

The following lemma is due to Govil, Rahman, and Schmeisser [4].

LEMMA 3. *If $f(z)$ is analytic in $|z| \leqslant 1$, $f(0) = a$, where $|a| < 1$, $f'(0) = b$, $|f(z)| \leqslant 1$ on $|z| = 1$, then, for $|z| \leqslant 1$,*

$$|f(z)| \leqslant \frac{(1-|a|)|z|^2 + |b||z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b||z| + (1-|a|)}.$$

The example $f(z) = (a + (b/(1+a))z - z^2)/(1 - (b/(1+a))z - az^2)$ shows that the estimate is sharp.

One gets easily from Lemma 3, the following

LEMMA 4. *If $f(z)$ is analytic in $|z| \leqslant R$, $f(0) = 0$, $f'(0) = b$, and $|f(z)| \leqslant M$ for $|z| = R$, then, for $|z| \leqslant R$,*

$$|f(z)| \leqslant \frac{M|z|}{R^2} \frac{M|z| + R^2|b|}{M + |z||b|}.$$

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. Consider

$$\begin{aligned} g(z) = (1-z)\,p(z) &= -a_n z^{n+1} + \sum_{k=1}^{n} (a_k - a_{k-1})\, z^k + a_0 \\ &= -a_n z^{n+1} + P(z), \qquad \text{say.} \end{aligned} \tag{3.1}$$

If $T(z)$ denotes the polynomial $\sum_{k=1}^{n} (a_k - a_{k-1})\, z^{n-k} + a_0 z^n$, then $T(z) = z^n P(1/z)$ and for $|z| = 1$, we have

$$\begin{aligned} |T(z)| &\leqslant \sum_{k=1}^{n} |a_k - a_{k-1}| + |a_0| \\ &\leqslant \sum_{k=1}^{n} (|a_k| - |a_{k-1}|) \cos \alpha \\ &\quad + \sum_{k=1}^{n} (|a_k| + |a_{k-1}|) \sin \alpha + |a_0| \qquad \text{(by Lemma 1)} \end{aligned}$$

$$= | a_n | (\cos \alpha + \sin \alpha) + 2 \sin \alpha \sum_{k=0}^{n-1} | a_k |$$
$$- | a_0 | (\cos \alpha + \sin \alpha - 1)$$
$$\leqslant | a_n | (\cos \alpha + \sin \alpha) + 2 \sin \alpha \sum_{k=0}^{n-1} | a_k |$$
$$= M_1 .$$

Hence, by the maximum modulus principle,

$$| T(0)| = | a_n - a_{n-1} | < M_1 .$$

Applying Lemma 2 to the function $T(z)$, we get for $| z | \leqslant 1$,

$$| T(z)| \leqslant M_1 \frac{M_1 | z | + | a_n - a_{n-1} |}{| a_n - a_{n-1} | | z | + M_1},$$

which implies that

$$\left| z^n P\left(\frac{1}{z}\right)\right| \leqslant M_1 \frac{M_1 | z | + | a_n - a_{n-1} |}{| a_n - a_{n-1} | | z | + M_1}, \qquad | z | \leqslant 1. \tag{3.2}$$

If $R > 1$, then $(1/R) e^{-i\theta}$ lies inside the unit circle for every real θ, and from (3.2) it follows that

$$| P(Re^{i\theta})| \leqslant M_1 R^n \frac{M_1 + | a_n - a_{n-1} | R}{| a_n - a_{n-1} | + M_1 R}, \tag{3.3}$$

for every $R \geqslant 1$ and θ real.

Thus for $| z | = R > 1$,

$$| g(Re^{i\theta})| \geqslant | -a_n R^{n+1} e^{i(n+1)\theta} + P(Re^{i\theta})|$$
$$\geqslant | a_n | R^{n+1} - | P(Re^{i\theta})|$$
$$\geqslant | a_n | R^{n+1} - M_1 R^n \frac{M_1 + R | a_n - a_{n-1} |}{M_1 R + | a_n - a_{n-1} |} \qquad \text{(by (3.3))}$$
$$\geqslant | a_n | R^{n+1} - M_1 R^n \frac{M_1 + cR}{M_1 R + c} \qquad \text{(by 1.3))}$$
$$= \frac{R^n}{M_1 R + c} [M_1 | a_n | R^2 - cR(M_1 - | a_n |) - M_1{}^2]$$
$$> 0,$$

if

$$R > \frac{c}{2}\left(\frac{1}{| a_n |} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{| a_n |} - \frac{1}{M_1}\right)^2 + \frac{M_1}{| a_n |}\right\}^{1/2}$$
$$= R_2 .$$

Therefore $p(z)$ has all its zeros in

$$|z| \leqslant R_2. \tag{3.4}$$

Next we show that $p(z)$ has no zeros in $|z| < R_3$. We have by (3.1)

$$\begin{aligned} g(z) &= a_0 + \sum_{k=1}^{n} (a_k - a_{k-1}) z^k - a_n z^{n+1} \\ &= a_0 + f(z), \qquad \text{say.} \end{aligned} \tag{3.5}$$

Let

$$M(R_2) = \operatorname*{Max}_{|z|=R_2} |f(z)|.$$

Since $R_2 \geqslant 1$ and $f(1) = -a_0$, we have $M(R_2) \geqslant |a_0|$.

Clearly

$$|f(z)| \leqslant |a_n| |z|^{n+1} + \sum_{k=1}^{n} |a_k - a_{k-1}| |z|^k,$$

and hence

$$\begin{aligned} M(R_2) &= \operatorname*{Max}_{|z|=R_2} |f(z)| \\ &\leqslant |a_n| R_2^{n+1} + R_2{}^n \sum_{k=1}^{n} |a_k - a_{k-1}| \\ &\leqslant |a_n| R_2^{n+1} + R_2{}^n \Big[\sum_{k=1}^{n} (|a_k| - |a_{k-1}|) \cos\alpha \\ &\qquad + (|a_k| + |a_{k-1}|) \sin\alpha \Big] \qquad \text{(by Lemma 1)} \\ &= |a_n| R_2^{n+1} + R_2{}^n \Big[|a_n| (\cos\alpha + \sin\alpha) \\ &\qquad + 2 \sum_{k=0}^{n-1} |a_k| \sin\alpha - |a_0| (\cos\alpha + \sin\alpha) \Big] \\ &= |a_n| R_2^{n+1} + |a_n| R_2{}^n \left[R - \frac{|a_0|}{|a_n|} (\cos\alpha + \sin\alpha) \right] \\ &= |a_n| R_2{}^n \left[R_2 + R - \frac{|a_0|}{|a_n|} (\cos\alpha + \sin\alpha) \right] \\ &= M_2, \qquad \text{say.} \end{aligned} \tag{3.6}$$

Further, because $f(0) = 0$, $f'(0) = a_1 - a_0 = b$, we have by Lemma 4,

$$|f(z)| \leqslant \frac{M_2 |z|}{R_2{}^2} \cdot \frac{M_2 |z| + R_2{}^2 |b|}{M_2 + |b| |z|} \tag{3.7}$$

for $|z| \leqslant R_2$.

Combining (3.5) and (3.7), we get, for $|z| \leqslant R_2$,

$$|g(z)| \geqslant |a_0| - \frac{M_2|z|}{R_2^2} \cdot \frac{M_2|z| + R_2^2|b|}{M_2 + |z||b|}$$
$$= \frac{-1}{R_2^2(M_2 + |z||b|)} [|z|^2 M_2^2 + R_2^2 |b||z| (M_2 - |a_0|)$$
$$- |a_0| R_2^2 M_2]$$
$$> 0,$$

if

$$|z| < \frac{-R_2^2|b|(M_2 - |a_0|) + \{R_2^4|b|^2(M_2 - |a_0|)^2 + 4|a_0| R_2^2 M_2^3\}^{1/2}}{2M_2^2}$$
$$= R_3,$$

which implies that $p(z)$ has no zeros in

$$|z| < R_3, \tag{3.8}$$

and the theorem follows.

Proof of Theorem 2. Again let

$$g(z) = (1 - z)p(z) = -a_n z^{n+1} + \sum_{k=1}^{n} (a_k - a_{k-1}) z^k + a_0$$
$$= -a_n z^{n+1} + P(z), \qquad \text{say}, \tag{3.9}$$

and

$$T(z) = z^n P\left(\frac{1}{z}\right) = \sum_{k=1}^{n} z^{n-k}(a_k - a_{k-1}) + a_0 z^n.$$

Then for $|z| = 1$, we have

$$|T(z)| \leqslant \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) + \sum_{k=1}^{n} (|\beta_{k-1}| + |\beta_k|) + \alpha_0 + |\beta_0|$$
$$= \left[\alpha_n + \left(2\sum_{k=0}^{n-1} |\beta_k| + |\beta_n|\right)\right]$$
$$= M_3, \qquad \text{say}.$$

Hence by the maximum modulus principle,

$$|T(0)| = |a_n - a_{n-1}| < M_3.$$

Therefore for $|z| \leqslant 1$, we have by Lemma 2

$$|T(z)| \leqslant M_3 \frac{M_3|z| + |a_n - a_{n-1}|}{|a_n - a_{n-1}||z| + M_3},$$

which implies

$$\left| z^n P\left(\frac{1}{z}\right)\right| \leqslant M_3 \frac{M_3 \mid z \mid + \mid a_n - a_{n-1} \mid}{\mid a_n - a_{n-1} \mid \mid z \mid + M_3}, \qquad \mid z \mid \leqslant 1. \tag{3.10}$$

If $R > 1$, then $(1/R)\, e^{-i\theta}$ lies inside the unit circle for every real θ and from (3.10) it follows that

$$\mid P(Re^{i\theta}) \mid \leqslant M_3 R^n \frac{M_3 + \mid a_n - a_{n-1} \mid R}{\mid a_n - a_{n-1} \mid + M_3 R}, \tag{3.11}$$

for every $R \geqslant 1$ and θ real. Thus for $\mid z \mid = R > 1$,

$$\begin{aligned}
\mid g(Re^{i\theta}) \mid &\geqslant \mid -a_n R^{n+1} e^{i(n+1)\theta} + P(Re^{i\theta}) \mid \\
&\geqslant \mid a_n \mid R^{n+1} - \mid P(Re^{i\theta}) \mid \\
&\geqslant \mid a_n \mid R^{n+1} - M_3 R^n \frac{M_3 + \mid a_n - a_{n-1} \mid R}{M_3 R + \mid a_n - a_{n-1} \mid} \qquad \text{(by (3.11))} \\
&= \mid a_n \mid R^{n+1} - M_3 R^n \frac{M_3 + cR}{M_3 R + c} \qquad \text{(by (1.3))} \\
&\geqslant \alpha_n R^{n+1} - M_3 R^n \frac{M_3 + cR}{M_3 R + c} \\
&= \frac{R^n}{M_3 R + c} [M_3 \alpha_n R^2 - cR(M_3 - \alpha_n) - M_3^2] \\
&> 0,
\end{aligned}$$

if

$$\begin{aligned}
R &> \frac{c}{2}\left(\frac{1}{\alpha_n} - \frac{1}{M_3}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{\alpha_n} - \frac{1}{M_3}\right)^2 + \frac{M_3}{\alpha_n}\right\}^{1/2} \\
&= R_4,
\end{aligned}$$

which implies that $p(z)$ has all its zeros in

$$\mid z \mid \leqslant R_4.$$

Next we show that $p(z)$ has no zeros in $\mid z \mid < R_5$. For this, we have by (3.9),

$$\begin{aligned}
g(z) &= a_0 + \sum_{k=1}^{n} (a_k - a_{k-1})\, z^k - a_n z^{n+1} \\
&= a_0 + f(z), \qquad \text{say.}
\end{aligned}$$

Since $R_4 \geqslant 1$, we have obviously $M(R_4) = \operatorname{Max}_{|z|=R_4} |f(z)| \geqslant |a_0|$, and

$$\begin{aligned}
M(R_4) &= \operatorname{Max}_{|z|=R_4} |f(z)| \\
&\leqslant |a_n| R_4^{n+1} + \sum_{k=1}^{n} |a_k - a_{k-1}| R_4^k \\
&\leqslant |a_n| R_4^{n+1} + R_4^n \sum_{k=1}^{n} |a_k - a_{k-1}| \\
&\leqslant |a_n| R_4^{n+1} + R_4^n \left[\sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) + \sum_{k=1}^{n} (|\beta_k| + |\beta_{k-1}|)\right] \\
&\leqslant (\alpha_n + |\beta_n|) R_4^{n+1} + R_4^n \left[\alpha_n - \alpha_0 + \left(2 \sum_{k=0}^{n-1} |\beta_k| + |\beta_n|\right) - |\beta_0|\right] \\
&= R_4^n[\alpha_n R_1 + (\alpha_n + |\beta_n|) R_4 - (\alpha_0 + |\beta_0|)] \\
&= M_4, \qquad \text{say.}
\end{aligned}$$

Lemma 4 and the lines of proof of Theorem 1 yield a proof of Theorem 2.

References

1. G. T. Cargo and O. Shisha, Zeros of polynomials and fractional order differences of their coefficients, *J. Math. Anal. Appl.* **7** (1963), 176–182.
2. N. K. Govil and V. K. Jain, On the Eneström–Kakeya theorem, *Ann. Univ. Mariae Curie-Sklodowska, Sect. A* **27** (1973), 13–18.
3. N. K. Govil and Q. I. Rahman, On the Eneström-Kakeya theorem *Tôhoku Math. J.* **20** (1968), 126–136.
4. N. K. Govil, Q. I. Rahman, and G. Schmeisser, On the derivative of polynomial, to appear.
5. A. Joyal, G. Labelle, and Q. I. Rahman, On the location of zeros of polynomials, *Canad. Math. Bull.* **10** (1967), 53–63.
6. P. V. Krishnaiah, On Kakeya's Theorem, *J. London Math. Soc.* **30** (1955), 314–319.